

Multiple Choice

1) (D) 80, angle at centre is twice angle at circumference

2) (A)

3) (C), $\left(2x - \frac{5}{x^3}\right)^{12} = \frac{(2x^4 - 5)^{12}}{x^{36}}$, $\therefore {}^{12}C_9 2^9 (-5)^3$

4) (D), $\tan \theta = \frac{-1-3}{1+(-1)(3)} = \frac{-4}{-2} = 2$

5) (B), $\prod \alpha = -42, \sum \alpha\beta = -41$

6) (B)

7) (A) $6 = \frac{2\pi}{n}, \therefore n = \frac{\pi}{3}, \therefore x = 5 \sin \frac{\pi t}{3}, \therefore v = \frac{5\pi}{3} \cos \frac{\pi t}{3}$

8) (D), ${}^{15}C_6 5! = \frac{15!5!}{6!9!} = \frac{15!}{6 \times 9!}$

9) (C), $x^4 - 8x^3 - 7x^2 + 3 = x(x+1)Q(x) + ax + 3$

Put $x = -1, 1 + 8 - 7 + 3 = -a + 3, \therefore a = -2$

10) (C), it is the perpendicular bisector of the 2 points,

$$m_1 = \frac{b+a-(b-a)}{a-b-(a+b)} = -\frac{a}{b}, \therefore m_2 = \frac{b}{a}$$

$$\text{Midpoint} = \left(\frac{a+b+a-b}{2}, \frac{b-a+b+a}{2} \right) = (a, b)$$

$$y - b = \frac{b}{a}(x - a), \therefore ay = bx$$

Question 11

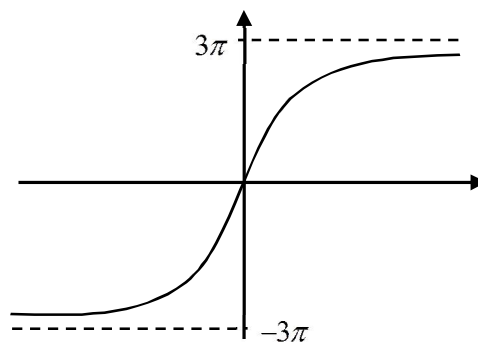
(a) Let $x + \frac{2}{x} = u, u^2 - 6u + 9 = (u-3)^2 = 0,$

$$\therefore u = 3, \therefore x + \frac{2}{x} = 3, \therefore x^2 - 3x + 2 = (x-1)(x-2) = 0$$

$$\therefore x = 1, 2$$

(b) $\text{Pr} = 0.9^{30} + {}^{30}C_1 0.1 \times 0.9^{29} + {}^{30}C_2 0.1^2 \times 0.9^{28}$

(c)



(d) $\int_2^5 \frac{x}{\sqrt{x-1}} dx$

Let $x = u^2 + 1, dx = 2udu.$

When $x = 2, u = 1$; when $x = 5, u = 2$

$$\int_1^2 \frac{u^2+1}{u} 2udu = 2 \int_1^2 (u^2+1) du = 2 \left[\frac{u^3}{3} + u \right]_1^2$$

$$= 2 \left(\frac{8}{3} - \frac{1}{3} \right) + 2(2-1) = \frac{14}{3} + 2 = \frac{20}{3}$$

(e) $\frac{x^2+5}{x} > 6, \therefore x^3 + 5x > 6x^2, \therefore x^3 - 6x^2 + 5x > 0$

$$x(x^2 - 6x + 5) = x(x-1)(x-5) > 0$$

$$\therefore 0 < x < 1, x > 5$$

(f) $y = \frac{e^x \ln x}{x},$

$$y' = \frac{e^x \left(\ln x + \frac{1}{x} \right) x - e^x \ln x}{x^2} = \frac{(x-1)e^x \ln x + e^x}{x^2}$$

Question 12

(a) (i) 4 m

(ii) $v = 6 \cos 3t, a = -18 \sin 3t$.

\therefore When it's first at rest, $\cos 3t = 0, \sin 3t = 1$,

$\therefore a = -18 \text{ m/s}^2$

(b) $V = \pi \int_0^{\frac{\pi}{8}} \cos^2 4x \, dx = \frac{\pi}{2} \int_0^{\frac{\pi}{8}} (1 + \cos 8x) \, dx$

$$= \frac{\pi}{2} \left[x + \frac{\sin 8x}{8} \right]_0^{\frac{\pi}{8}} = \frac{\pi}{2} \left(\frac{\pi}{8} \right) = \frac{\pi^2}{16}$$

(c) $\ddot{x} = 2 - e^{-\frac{x}{2}} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$

$$\frac{1}{2} v^2 = 2x + 2e^{-\frac{x}{2}} + C$$

$v = 4, x = 0, \therefore 8 = 2 + C, \therefore C = 6$

$$\therefore v^2 = 2 \left(2x + 2e^{-\frac{x}{2}} + 6 \right) = 4 \left(x + e^{-\frac{x}{2}} + 3 \right)$$

(d) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$

Let $x = -1, 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n}$

(e) The tangent to the curve at $x = x_1$ meets the x -axis at a point further to α than x_1 is.

(f) $T = 23 - Be^{-0.03t}$

When $t = 0, T = 2, \therefore 2 = 23 - B, \therefore B = 21$

$\therefore T = 23 - 21e^{-0.03t}$

When $T = 10, 10 = 23 - 21e^{-0.03t}$

$$e^{-0.03t} = \frac{13}{21}$$

$$-0.03t = \ln \frac{13}{21}$$

$$\therefore t = \frac{\ln \frac{13}{21}}{-0.03} \approx 16 \text{ minutes.}$$

Question 13

(a) Let $n = 1, 2 + 1 = 3$ is divisible by 3, \therefore true.

Assume $2^n + (-1)^{n+1} = 3M$, where M is an integer

$$\therefore 2^n = 3M - (-1)^{n+1} = 3M + (-1)^{n+2}$$

Required to prove that $2^{n+1} + (-1)^{n+2}$ is divisible by 3.

$$2^{n+1} + (-1)^{n+2} = 2(3M + (-1)^{n+2}) + (-1)^{n+2}$$

$$= 6M + 3(-1)^{n+2}$$

$$= 3(2M + (-1)^{n+2}), \text{ which is divisible by 3.}$$

$\therefore 2^n + (-1)^{n+1}$ is divisible by 3 for all $n \geq 1$.

(b) (i) $L^2 = 40^2 + x^2$

$$2L \frac{dL}{dx} = 2x$$

$$\frac{dL}{dx} = \frac{x}{L} = \cos \theta$$

(ii) $\frac{dL}{dt} = \frac{dx}{dt} \frac{dL}{dx} = 3 \cos \theta$

(c) (i) $P(2at, at^2), S(0, a)$

$$Q \begin{cases} x = \frac{1 \times 2at + t^2 \times 0}{1 + t^2} = \frac{2at}{1 + t^2} \\ y = \frac{1 \times at^2 + t^2 \times a}{1 + t^2} = \frac{2at^2}{1 + t^2} \end{cases}$$

(ii) $m_{OQ} = \frac{\frac{2at^2}{1+t^2}}{\frac{2at}{1+t^2}} = t$

(iii) $m_P = \frac{dy/dt}{dx/dt} = \frac{2at}{2a} = t$

Since the gradient of the tangent at P is also t , $OQ \parallel$ the tangent at P .

But from the optical property of the parabola, $SP = ST$, where T is the point the tangent at P meets the y -axis, $\therefore \Delta SPT \parallel \Delta SQO, \therefore SQ = SO = a$.

\therefore The locus of Q is a circle of centre S , radius a .

(d) (i) In a cyclic quad, the interior angle is equal to the opposite exterior angle

(ii) $\angle OPA = \angle OAP$ ($OP = OA =$ radius, $\therefore \Delta OPA$ is isosceles)

Extend OP to $Q, \angle OPA = \angle CPQ$ (vertically opposite)

$\therefore \angle CPQ = \angle CQP$ (since both $= \angle OAP$)

$\therefore OP$ is a tangent to the circle CPQ (angles in alternate segments are equal)

Question 14

$$(a) (i) x = Vt \cos \theta, \therefore t = \frac{x}{V \cos \theta}$$

$$\begin{aligned} \therefore y &= -\frac{1}{2}g \frac{x^2}{V^2 \cos^2 \theta} + V \frac{x}{V \cos \theta} \sin \theta \\ &= x \tan \theta - \frac{gx^2}{2V^2 \cos^2 \theta} \\ &= x \tan \theta - \frac{gx^2}{2V^2} \sec^2 \theta \end{aligned}$$

(ii) P is the point on the line $y = -x$

$$-x = x \tan \theta - \frac{gx^2}{2V^2} \sec^2 \theta$$

$$-1 = \tan \theta - \frac{gx}{2V^2} \sec^2 \theta$$

$$\frac{gx}{2V^2} \sec^2 \theta = 1 + \tan \theta$$

$$\begin{aligned} \therefore x &= \frac{2V^2}{g} \frac{1 + \tan \theta}{\sec^2 \theta} = \frac{2V^2}{g} (\cos^2 \theta + \sin \theta \cos \theta) \\ &= \frac{2V^2}{g} \cos \theta (\cos \theta + \sin \theta) \end{aligned}$$

$$\therefore D = \sqrt{2}x = \frac{2\sqrt{2}V^2}{g} \cos \theta (\cos \theta + \sin \theta)$$

$$(iii) D = \frac{2\sqrt{2}V^2}{g} (\cos^2 \theta + \sin \theta \cos \theta)$$

$$= \frac{2\sqrt{2}V^2}{g} \left(\frac{1 + \cos 2\theta}{2} + \frac{\sin 2\theta}{2} \right)$$

$$\frac{dD}{d\theta} = \frac{2\sqrt{2}V^2}{g} (-\sin 2\theta + \cos 2\theta)$$

$$(iv) \frac{dD}{d\theta} = 0 \text{ when } \sin 2\theta = \cos 2\theta, \therefore \tan 2\theta = 1$$

$$\therefore 2\theta = \frac{\pi}{4}, \therefore \theta = \frac{\pi}{8}$$

$$\frac{d^2D}{d\theta^2} = \frac{4\sqrt{2}V^2}{g} (-\cos 2\theta - \sin 2\theta)$$

$$= \frac{4\sqrt{2}V^2}{g} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) = -\frac{8V^2}{g} < 0 \text{ when } \theta = \frac{\pi}{8}$$

$$\therefore D \text{ is maximum when } \theta = \frac{\pi}{8}$$

(b) (i) Player A takes the first turn.

If the arrow stops in section P , he wins, $\text{Pr} = p$

If the arrow stops in section R , and player B 's

arrow stops in section Q , player A wins, $\text{Pr} = rq$

$$\therefore \text{Pr} = p + rq$$

$$= p + r(1 - p - r), \text{ since } p + q + r = 1$$

$$= p + r - rp - r^2$$

$$= (p + r) - r(p + r)$$

$$= (1 - r)(p + r)$$

(ii) The probabilities that player A wins in the first,

second, third, and fourth turns are p, rq, r^2p, r^3q, \dots

\therefore The probability that he eventually wins the game is

$$p + rq + r^2p + r^3q + r^4p + r^5q + \dots$$

$$= (1 - r)(p + r) + r^2(1 - r)(p + r) + r^4(1 - r)(p + r) + \dots,$$

$$\text{since } p + rq = (1 - r)(p + r)$$

$$= (1 - r)(p + r)(1 + r^2 + r^4 + \dots)$$

$$= (1 - r)(p + r) \frac{1}{1 - r^2}$$

$$= \frac{p + r}{1 + r}$$